CHAPTER THREE
OPTIMIZATION TECHNIQUES

3.1 Introduction

An optimization technique is a technique of maximizing or minimizing a function. In simple words, it is a technique of finding the value of the independent variable(s) that maximizes or minimizes the value of the dependent variable. For example, some firms may be interested in finding the level of output that maximizes their total revenue; some firms facing a constant price may want to find the level of output that would minimize the average cost; and most important of all, most firms may be interested in finding the level of output that maximizes their profits.

Differential calculus and optimization

3.2 The Rules of Differentiation

The nature of functions that are encountered in managerial decisions are (*i*) function of a constant, (*ii*) power function, (*iii*) function as a sum or difference of two functions, (*iv*) function as a product of two functions, (*v*) function of a function. For describing the rules of differentiation, we will use the alphabet Y as the dependent variable, alphabet X as the independent variable, and alphabets a, b, and c as constraints.

1. Derivative of a Constant Function

The derivative of a constant function equals zero. For example,

if $Y = f(X) = a$ (where a is constant) .... (3.2.1)

then $$\frac{\delta Y}{\delta X} = 0$$

The reason is that a constant function implies that whatever the value of X, the value of Y remains constant. That is, even if the value of X changes, the value of Y does not change. For example, if the optimum level of capital-labor ratio has been reached and capital is constant, then the production function can be expressed as

$$Y = f(X) = 500$$

where Y is output and X is labor. Given this function, output will remain constant whatever the number of workers employed.

2. Derivative of a Power Function

A power function takes the following form.

$$Y = f(X) = aX^b \text{ .... } (3.2.2)$$

where a and b are constants, a being the coefficient of X and b power of X.

The derivative of Y with respect to X is power b times a times X raised to the power $b - 1$.

That is,

$$\frac{\delta Y}{\delta X} = baX^{b-1} \text{ .... } (3.2.3)$$

Example:

(a) If $Y = 5X^3$

then $\frac{\delta Y}{\delta X} = 3 * 5 * X^{3-1} = 15X^2$

(b) If $Y = 4X^2$

then $\frac{\delta Y}{\delta X} = 2 * 4 * X^{2-1} = 8X$

(c) If $Y = 2X$

then $\frac{\delta Y}{\delta X} = 1 * 2 * X^{1-1} = 2X^0 = 2$

(d) If $Y = X$

then $\frac{\delta Y}{\delta X} = X^{1-1} = 1$

3. Derivative of Functions of Sum and Difference of Functions

A dependent variable Y may be the function of the sum (or difference) of two different functions of the same independent variable X or of a sum (or difference) of two other variables which are functions of X. The derivatives of such functions are given below.

$$Y = f(X) + g(X)$$

where $f(X)$ and $g(X)$ denote two different functional relationships between Y and X.

The derivative function can be expressed as

$$\frac{\delta Y}{\delta X} = \frac{\delta f(X)}{\delta X} + \frac{\delta g(X)}{\delta X}$$

And if $Y = f(X) - g(X)$

(where $f(X)$ and $g(X)$ denote two different functions)

then

$$\frac{\delta Y}{\delta X} = \frac{\delta f(X)}{\delta X} - \frac{\delta g(X)}{\delta X}$$

Example:

(*i*) If $Y = 5X + 2X^3$

then $\frac{\delta Y}{\delta X} = 5X^{1-1} + 2 * 3\,X^{3-1} = 5 + 6X^2$

(*ii*) If $Y = 5X^2 - 2X^4$

then $\frac{\delta Y}{\delta X} = 2 * 5\,X^{2-1} - 4 * 2\,X^{4-1} = 10X - 8X^3$

(*iii*) If $Y = 4X^3 - 3X^2 + 3$

then $\frac{\delta Y}{\delta X} = 3 * 4\,X^{3-1} - 2 * 3\,X^{2-1} + 0 = 12X^2 - 6X$

4. Derivative of a Function as a Product of Two Functions

The derivative of a function as a product of two functions is equal to the first term (or function) multiplied by derivative of the second function plus second term (or function) multiplied by the derivative of the first function. For example, suppose *Y* is the function of two different functions of the same independent variable, *X*, i.e.,

$$Y = f(X) \text{ x } g(X)$$

where *f(X)* and *g(X)* denote two different functional relationships between *Y* and *X*.

The derivative function can be expressed as

$$\frac{\delta Y}{\delta X} = f(X) \text{ x } \frac{\delta\, g(X)}{\delta X} + g(X) \text{ x } \frac{\delta\, f(X)}{\delta X}$$

5. Derivative a Quotient

If a function is in the form of a *quotient*, then *the derivative of the function is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator*, whole divided by the square of the denominator. For example, suppose

$$Y = \frac{f(X)}{g(X)} \text{ .... (3.2.4)}$$

Then

$$\frac{\delta Y}{\delta X} = \frac{g(X) \text{ x } \frac{\delta\, f(X)}{\delta X} - f(X) \text{ x } \frac{\delta\, g(X)}{\delta X}}{[g\,(X)^2} \text{ .... (3.2.45}$$

Consider another example. Suppose

$$Y = \frac{5X + 4}{2X + 3}$$

Then the derivative of the function is given as

$$\frac{\delta Y}{\delta X} = \frac{(2X + 3)(5) - (5X + 4)(2)}{(2X + 3)^2}$$

$$= \frac{(10X + 15) - (10X + 8)}{(2X + 3)^2}$$

$$= \frac{7}{(2X + 3)^2}$$

6. Chain Rule

If Y is a function of Z and again Z is a function of X, then

$$\frac{\delta Y}{\delta X} = \frac{\delta Y}{\delta Z} \times \frac{\delta Z}{\delta X} = \frac{\delta Y}{\delta X}$$

$Y = 5Z + 2Z^2$ and $Z = 2X + 10X^2$

Then $\frac{\delta Y}{\delta z} = 5 + 4z$ *and* $\frac{\delta Z}{\delta X} = 2 + 20X$

Now $\frac{\delta y}{\delta x} = (5+4U)(2+20X)$

3.2.1 The Partial Derivative

Functions with Several Independent Variables: Many functions used in economic and business analysis have more than one independent variable. Some common examples of such functions are demand function, production function and cost function.

Demand function: $D_x = f\,(P_x, P_s, P_c, R_d, A, T, \ldots \text{etc.})$

where, D_x = demand for commodity *X*,

P_x = price of *X*,

P_s = price of substitutes,

P_c = price of complements,

R_d = disposable resources (income),

A = advertisement of expenditure by producers,

T = tastes and preferences etc.

Production function: $Q = f\,(K, L)$

where, Q = quantity produced,

K = quantity of capital,

L = number of workers.

Cost function: $C = f\,(K, r, L, w)$

where, C = total cost,

K = capital,
r = rental rent,
L = number of workers,
w = wage rate.

Rules of Partial Differentiation: We will describe the rules of partial differentiation in terms of *Y*, as dependent variable and *X* and Z as two independent variables. Suppose $Y = f(X, Z)$ and the functional relationship between *Y* (the dependent variable) and the independent variables, X and Z, is given as

$$Y = X^3 + 4XZ + 5Z^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.2.4)$$

The rules of partial differentiation can be stated as follows:

(*i*) Only one of the independent variables is allowed to change at a time and all other independent variables are held constant.

(*ii*) For differentiating the dependent variable with respect to one independent variable, the rule of differentiation is followed.

Based on these rules, the derivatives of *Y* with respect to *X* and Z in equation (3.2.4) are given below.

(*i*) Derivative of *Y* with respect to *X* with Z held constant,

$$\frac{\delta Y}{\delta X} = 3X^{3-1} + 4Z = 3X^2 + 4Z$$

(*ii*) Derivative of *Y* with respect to Z with *X* remaining constant,

$$\frac{\delta Y}{\delta Z} = 4X + 2 \, x \, 5Z^{2-1} = 4X + 10Z$$

In case a function with two (or more) independent variables is in the multiplier form such as

$$Y = aX^b Z^c$$

then the derivative of *Y* with respect to *X* is

$$\frac{\delta Y}{\delta X} = baX^{b-1} Z^c$$

and the derivative of *Y* with respect to Z is

$$\frac{\delta Y}{\delta Z} = aX^b cZ^{c-1} = acX^b Z^{c-1}$$

Types of optimization techniques

3.3 Technique of Maximizing Total Revenue

The total revenue *(TR)* of a firm is defined as: $TR = P.Q$ .... *(3.1)*

Where *P* = price and *Q* = quantity sold.

Suppose a price function is given as $P = 500 - 5Q$ .... *(3.2)*

By substituting equation (3.2) into equation (3.1), we get *TR* as follows.

$TR = (500 - 5Q)Q$

$= 500Q - 5Q^2$ .... (3.3)

Now the problem is to find the value *of Q* that maximizes total revenue.

3.3.1 The Rule of Total Revenue Maximization

The rule of maximization of total sales revenue is that, the total revenue is maximum at the level of sales *(Q)* at which ***MR* = 0,** that is, the marginal revenue *(MR)*, i.e., the revenue from the sale of the marginal unit of the product, must be equal to zero. *MR* is given by the first derivative of the *TR* function. So, to find the value *Q* that maximizes *TR*, we need to find the derivative of the *TR* function (3.3) with respect to *Q;* set it equal to zero and solve it for *Q*, as shown below. Given the *TR* function in Equation (3.3), the first derivative of the *TR function* can be obtained as follows.

$\frac{\delta TR}{\delta Q} = 500 - 10Q$ .... (3.4)

By setting equation (3.4) equal to zero and solving for Q, we get

$500 - 10Q = 0$

$-10Q = -500$

$\underline{\underline{Q = 50}}$ .... (3.5)

Equation (3.5) shows that $Q = 50$ maximizes the total revenue.

The maximum TR can be obtained by substituting 50 for *Q* in the TR function (3.3). Thus,

$TR = 500 \times (50) - 5(50)^2$ .... (3.6)

$= 25{,}000 - 12{,}500$

$= \underline{\underline{\mathbf{12{,}500}}}$

Let us now check the result. Whether TR = Rs. 12,500 is maximum can be checked by increasing and decreasing *Q* by one unit and then comparing TR at $Q = 51$ and at $Q = 49$ with TR at $Q = 50$.

TR (at $Q = 51$) $= 500(51) - 5(51)^2$

$= 25{,}500 - 13{,}005$

$= \underline{\underline{\mathbf{12{,}495}}}$

TR (at $Q = 49$) $= 500(49) - 5(49)^2$

$= 24{,}500 - 12{,}005$

$= \underline{\underline{\mathbf{12{,}495}}}$

The calculation made above show that if sales are increased above 50 units or reduced below 50 units, TR decreases in both the cases. Thus, it is proved that $Q = 50$ maximizes TR.

3.4 Technique of Optimizing Output: Minimizing Average Cost

The optimum size of the firm is one that minimizes the average cost of production. It is also called the **most efficient size of the firm**. A prior knowledge of the optimum size of the firm is very important for future planning under at least three conditions.

One, a businessman planning to set up a new production unit would like to know the optimum size of the plant for future planning. This problem arises because, as the theory of production tells us, the advantage cost of production in most productive activities decreases to a certain level of output and then begins to increase.

Two, the firms planning to expand their scale of production would like to know the most efficient level of the economies of scale so that they are able to plan the marketing of the product accordingly.

Three, businessmen working in a competitive market are often faced with a given market price. Their profit then depends on their ability to reduce their unit cost of production. Given the technology and input prices, the prospect of reducing the unit cost of production depends invariably on the size or production. The problem that decision makers might face in this regard is how to find the *optimum level* of output, i.e., the level of output that minimizes the average of production.

As already mentioned, under the general production conditions, **the optimum level of output is the one that minimizes the average cost**. The average cost (AC) can be obtained by dividing total cost (TC) by the quantity produced (Q).

That is, $AC = \dfrac{TC}{Q}$ .... (3.7)

Suppose the TC function of a firm is given as

$$TC = 100 + 60Q + 4Q^2 \text{ .... (3.8)}$$

Then $AC = \dfrac{100 + 60Q + 4Q^2}{Q}$

$$= 100/Q + 60 + 4Q = 100Q^{-1} + 60 + 4Q \text{ .... (3.9)}$$

Now the problem is how to find the value of Q that minimizes AC.

The Rule of Minimization. Like the rule of maximization, *the rule of minimizing a function is that its derivative must be equal to zero*. So the value of *Q* that minimizes AC can be obtained by finding the derivative of the AC function and setting it equal to zero and solving it for *Q*.

The derivative of the AC function (3.9) is given as

$$\frac{\delta \text{ AC}}{\delta Q} = -100/Q^2 + 4 \text{ ....} \quad (3.10)$$

Be setting equation (3.10) equal to zero, we get

$$-100/Q^2 + 4 = 0$$
$$-100/Q^2 = -4$$
$$Q^2 = -100/-4 = 25 \text{ ....} \quad (3.11)$$
$$\underline{\underline{Q = 5}}$$

The result shows that *Q* = 5 minimizes the average cost. In other words, the optimum size of the output is 5 units. Any other output will increase the average cost of production.

3.5 Maximization of Profit

Profit maximization is the most common objective of business firms.

$$\pi = \text{TR} - \text{TC} \text{ ....} \quad (3.12)$$

Total profit (π) is maximum when TR - TC is maximum. Therefore, a profit maximizing firms tries to maximize TR - TC. Recall that both TR and TC are positive functions of the same variable, *Q*. The problem that decision-makers face is 'how to determine the level of output (*Q*) **which maximizes profit'**. The technique of differentiation is of great help in finding the answer to this problem. There are two alternative ways of finding *Q* at which profit is maximum: (i) going by the rules of profit maximization, and (ii) maximizing the profit function.

3.5.1 Profit Maximization Conditions

There are two conditions of profit maximization: (i) the *necessary* or the *first order condition*, and (ii) the *supplementary* or the *second order condition*.

(i) The *necessary* or the *first order condition* requires that *MC* must be equal to *MR*. This means that for profit to be maximum;

$$MR = MC$$

The first order condition can be written as:

$$\frac{\delta\, TR}{\delta Q} = \frac{\delta\, TC}{\delta Q}$$

or

$$\frac{\delta\, TR}{\delta Q} - \frac{\delta\, TC}{\delta Q} = 0 \text{ ....} \quad (3.13)$$

It means that the *first derivative* of the TR *function* must be equal to the *first derivative* of the TC *function* or their difference must be equal to zero.

(ii) The *supplementary* or the *second order condition* requires that the difference between the *second derivative* of the *TR function* and the *second derivatives of the two functions must be equal*. Incidentally, ***the derivative of the first derivative* of a function is called *the second derivative***. The second order condition requires that

$$\frac{TR}{\delta Q^2} < \frac{\delta^2 TC}{\delta Q^2} \quad \text{or} \quad \frac{\delta^2 TR}{\delta Q^2} + \frac{\delta^2 TC}{\delta Q^2} < 0 \text{ ....} \quad (3.14)$$

$$\text{Or } \frac{\delta 2\pi}{\delta\, Q2} < 0$$

Now let us apply these conditions to the TR and TC functions and find (a) the profit maximizing output, (b) the maximum profit, and (c) proof that profit is maximum.

Suppose that the TR and TC functions are given, respectively, as follows.

$$TR = 600Q - 3Q^2 \text{ ....} \quad (3.15)$$

$$TC = 1000 + 100Q + 2Q^2 \text{ ....} \quad (3.16)$$

Given the TR and TC functions as in equation (3.15) and (3.16), respectively, MR and MC can be obtained as follows.

$$MR = \frac{\delta\, TR}{\delta Q} = 600 - 6Q \text{ ....} \quad (3.17)$$

and

$$MC = \frac{\delta\, TC}{\delta Q} = 100 + 4Q \text{ ....} \quad (3.18)$$

By applying the *first order condition* of profit maximization, we get maximum profit where

$$MR = MC$$

$$600 - 6Q = 100 + 4Q \qquad \text{...........................} \quad (3.19)$$

$$- 6Q - 4Q = - 600 + 100$$

$$- 10Q = - 500$$

$$\underline{\underline{Q = 50}}$$

The first order condition of profit maximization reveals that, given the TR and TC functions, the total profit is maximum at $Q = 50$.

Let us now apply the *second order* condition. Given the first order derivative of the TR function in equation (3.17) and that of the TC function in Equation (3.18), the *second derivatives* of the TR and TC functions are presented below.

$$\frac{\delta^2 TR}{\delta Q^2} = \frac{\delta\, MR}{\delta Q} = -6$$

and

$$\frac{\delta^2 TC}{\delta Q^2} = \frac{\delta\, MC}{\delta Q} = 4$$

Note that the second derivative of the TR function equals - 6 and the second derivative of the TC function equals 4. The sum of the two second derivatives, i.e., $-6 + 4 = -2$ and $-2 < 0$. So the second order condition of profit maximization is also satisfied at $Q = 50$.

Is Total Profit Maximum at $Q = 50$? This can be checked by comparing profits at $Q = 50$, at $Q > 50$ and $Q < 50$. By substituting these numbers by turn into profit function, we can get the total profit at three levels of output. Let us first work out the profit at $Q = 50$.

$$\begin{aligned}
\text{Total profit (at } Q = 50) &= TR - TC \\
&= (600Q - 3Q^2) - (1000 + 100Q + 2Q^2) \\
&= \{600\,(50) - 3\,(50)^2\} - \{1000 + 100\,(50) + 2\,(50)^2\} \\
&= 22{,}500 - 11{,}000 = \underline{\underline{\mathbf{11{,}500}}} \\
\text{Total profit (at } Q = 51) &= TR - TC \\
&= \{600\,(51) - 3\,(51)^2\} - \{1000 + 100\,(51) + 2\,(51)^2\} \\
&= 22{,}797 - 11{,}302 = \underline{\underline{\mathbf{11{,}495}}} \\
\text{Total profit (at } Q = 49) &= TR - TC \\
&= \{600\,(49) - 3\,(49)^2\} - \{1000 + 100\,(49) + 2\,(49)^2\} \\
&= 22{,}197 - 10{,}702 = \underline{\underline{\mathbf{11{,}495}}}
\end{aligned}$$

The foregoing calculations show that at Q = 50 profits equal Rs. 11,500. And, if Q is increased or decreased even by a single unit, the total profit decreases by Rs. 5 in either case. This proves that profit is maximized at 50 units of output.

3.5.2 Maximization of Profit Function

Profit function is given as $\pi = TR - TC$;

$$\begin{aligned}
\pi &= 600Q - 3Q^2 - (1000 + 100Q + 2Q^2) \\
&= 600Q - 3Q^2 - 1000 - 100Q - 2Q^2 \\
&= -1000 + 500Q - 5Q^2 \quad \text{..........................} \quad (3.20)
\end{aligned}$$

Going by the maximization rule, for profit to be maximum, the derivative of the profit function (3.20) must be equal to zero. The derivative of the profit function is

$$\frac{\delta \pi}{\delta Q} = 500 - 10Q \text{ ....} \quad (3.21)$$

For profit to be maximum, the first derivative of the profit function must be equal to zero.

That is,

$$500 - 10Q = 0$$

$$\underline{\underline{Q = 50}}$$

Note that the profit to be maximizing output obtained by the alternative methods is the same, i.e., Q = 50.

Optimization of a Multivariate Profit Function: We have so far discussed the optimization (maximization and minimization) of a function **with one independent variable.** Most decision makers deal with functions with more than one independent variable. For example, output is the function of two independent variable inputs, labor and capital; total revenue is not the function of quantity alone but also the advertisement expenditure; in case of firms producing more than one commodity that is, the multi-product firms, profit is function of all the products. In this section, we will explain the technique of optimization of a multivariate function assuming a simple case of independent variables. One common case is that of profit maximization by a firm which **produces two commodities**. We will, therefore, explain here the profit maximization technique assuming a case of two products. In this case, the problem is to find the outputs of both products that maximize the profit.

The profit function in the case of two products, say X and Y, can then be expressed as

$$\pi = f(X, Y)$$

Suppose that the profit function is given as follows.

$$\pi = 100X - 2X^2 - XY + 180Y - 4Y^2 \text{ ....} \quad (3.22)$$

Maximization of a multivariate profit function (3.22) requires that (*i*) *partial derivative of* π with respect to X and (*ii*) *partial derivative* of π with respect to Y are each set equal to zero and solved for X and Y.

The partial derivative of π with respect to X, holding Y constant, is

$$\frac{\delta \pi}{\delta X} = 100 - 4X - Y \text{ ....} \quad (3.23)$$

and partial derivative of π with respect to Y, holding X constant, is

$$\frac{\delta\pi}{\delta X} = 180 - X - 8Y \text{ ....} \quad (3.24)$$

Setting each of the partial derivative given in Equations (3.23) and (3.24) equal to zero, we get

(i) $100 - 4X - Y = 0$

(ii) $180 - X - 8Y = 0$

Note that setting each partial derivative equal to zero results in two simultaneous equations. By solving these equations we can find the values of X and Y that maximize the profit function. To solve the equations, we need to eliminate one of the variables (say, X). For this, let us multiply equation (ii) by 4 and subtract the product from equation (i). Thus, we have

(iii) $100 - 4X - Y = 0$ (equation (iii) is the same as equation (i))

(iv) $720 - 4X - 32Y = 0$

By subtracting equation (iv) from equation (iii), we get

$$-620 + 31Y = 0$$

$$\underline{\underline{Y = 20}}$$

By substituting 20 for Y in equation (i) we can obtain the value of X.

$$100 - 4X - 20 = 0$$

$$-4X = -80$$

$$\underline{\underline{X = 20}}$$

The forgoing calculations show that the firm can maximize its profit function by producing 20 units each of its products X and Y. The maximum profit can be worked out by substituting the values of X and Y in the profit function.

$$\pi = 100\,(20) - 2\,(20)^2 - (20)\,(20) + 180\,(20) - 4\,(20)^2$$

$$= 2000 - 800 - 400 + 3600 - 1600$$

$$= \underline{\underline{\mathbf{2{,}800}}}$$

Any other combination of X and Y will reduce the profit. Whether the profit is maximum can be checked by substituting any value other than 20 for X and Y.

3.6 Constrained Optimization

The maximization techniques discussed above can be called *unconstrained* optimization techniques, in the sense that they assume that firms operate under no *constraints* on their activity. For example, in the case output maximization, firms face no resource constraints: they possess unlimited resources and can acquire all the inputs, finance, capital equipment, men and raw materials that they need to maximize output. **Same is the case with cost *minimization* technique.** The firms have all the resources to carry out production activity until *average cost* is minimized or cost for a given output is minimized. In the real business world, however, the managers face serious resource *constraints*. For example, they need to maximize output with given quantity of capital and labor time. The technique that is used to optimize the business objective(s) under constraints are called *constrained optimization techniques*. There are three very common techniques of constrained optimization, *Linear programming*, *Constrained optimization by substitution* and *Lagrangian multiplier*. The linear programming technique has a wide range of application and is a subject in itself.

3.6.1 Constrained Optimization by Substitution Technique

(i) Constrained Profit Maximization: Let us recall our earlier example of profit maximization.

$$\pi = 100X - 2X^2 - XY + 180Y - 4Y^2 \text{ .... } (3.25)$$

We have illustrated above maximization of this profit function *without any constraint*. That is, the independent variables, X and Y, were free to take any value in the profit maximization solution.

Here we illustrate the maximization of the same profit function with a constraint on output that the sum of X and Y must be equal to 30 (instead of 40 as in the solution without constraints). That is,

$$X + Y = 30 \text{ .... } (3.26)$$

A constrained problem of this kind can be solved by *substituting method* as illustrated below. The process of solution involves two steps: (*i*) express one of the variables in terms of the other and solve the constraint equation for one of the variables (X or Y) and (*ii*) substitute the solution into the objective function to be maximized and solve it for the other variable.

Given the constraint equation (3.26), the values of X and Y can be expressed in terms of one another as follows.

$$X = 30 - Y$$

or

$$Y = 30 - X$$

We can now substitute the value of X (or Y), in to equation (3.25) and find the maximization solution. By substituting the value of X, the profit function (3.25) can be expressed as

$$\pi = 100\,(30 - Y) - 2\,(30 - Y)^2 - (30 - Y)\,Y + 180Y - 4Y^2$$

$$= 3000 - 100Y - 2\,(900 - 60Y + Y^2) - 30Y + Y^2 + 180Y - 4Y^2$$

$$= 3000 - 100Y - 1800 + 120Y - 2Y^2 - 30Y + Y^2 + 180Y - 4Y^2$$

$$= 1200 + 170Y - 5Y^2 \text{ .... } (3.27)$$

Note that the *substitution method* converts a constrained problem into an unconstrained one. Equation 3.27 can now be maximized by obtaining its derivative and setting it equal to zero and solving it or Y.

Thus, $$\frac{\delta \pi}{\delta Y} = 170 - 10Y \quad \text{.........................} \quad (3.28)$$

Be setting equation (3.28) equal to zero, we get

$$170 - 10Y = 0$$

$$\underline{\underline{Y = 17}}$$

By substituting 17 for Y in constant equation (3.26), we get

$$X + 17 = 30$$

$$\underline{\underline{X = 13}}$$

Thus, the optimal solution of the profit maximization problem is $X = 13$ and $Y = 17$. These values of X and Y satisfy the constraint. In simple words, we get the optimization solution that the firm maximizes its profit by producing 13 units of X and 17 units of Y. The answer will be the same if we substitute 30 - X for Y in equation (3.25) and solve the equation for X.

Now let us compute the maximized profit under constraints. This can be done by substituting the values of X and Y into the profit function (3.25). By substitution, we get

$$\pi = 100\,(13) - 2\,(13)^2 - (13)\,(17) + 180\,(17) - 4\,(17)^2$$

$$= \underline{\underline{\mathbf{2{,}645}}}$$

Note that maximum profit (2,645) under constraint is less than the maximum profit under no constraint (2800).

(ii) Constrained Cost Minimization: Let us now apply the *substitution method* of optimization to a problem of constrained cost minimization. Suppose that the cost function of a firm producing two goods, X and Y, is given as

$$TC = 2X^2 - XY + 3Y^2$$

and the firm has to meet a combined order of 36 units of the two goods. The manager's problem is to find an optimum combination of X and Y that minimizes the cost of production. The problem can be restated formally as

Minimize $$TC = 2X^2 - XY + 3Y^2 \quad \text{.........................} \quad (3.29)$$

Subject to: $$X + Y = 36 \quad \text{.........................} \quad (3.30)$$

Substitution method requires that the constraint equation (3.30) is expressed in terms of any one of the two goods and then substituted into the *objective function* (3.29). By expressing X in terms of Y, we get

$$X = 36 - Y \quad \text{..........................} \quad (3.31)$$

By substituting equation (3.31) for X in the objective function (3.29), we get

$$\begin{aligned} TC &= 2(36 - Y)^2 - (36 - Y)Y + 3Y^2 \\ &= 2(1296 - 72Y + Y^2) - 36Y + Y^2 + 3Y^2 \\ &= 2592 - 144Y + 2Y^2 - 36Y + Y^2 + 3Y^2 \\ &= 2592 - 180Y + 6Y^2 \quad \text{..........................} \quad (3.32) \end{aligned}$$

For the objective function (3.32) to be minimized, its first derivative must be set to zero. Thus,

$$\frac{\delta TC}{\delta Y} = -180 + 12Y = 0 \quad \text{..........................} \quad (3.33)$$

Solving equation (3.33) for Y, we get

$$12Y = 180$$

$$\underline{\underline{Y = 15}}$$

By substituting the value of Y in the constraint equation (3.30) we get

$$X + 15 = 36$$

$$\underline{\underline{X = 21}}$$

Thus, we get the optimum solution that $X = 21$ and $Y = 15$ minimize the cost of meeting the order. The minimum cost of producing 21 units of X and 15 units of Y can be obtained by substituting these values in cost function (3.29).

$$\begin{aligned} \text{Minimum cost} &= 2(21)^2 - (21)(15) + 3(15)^2 \\ &= 882 - 315 + 675 \\ &= \underline{\underline{\mathbf{1,242}}} \end{aligned}$$

3.6.2 Constrained Optimization by Lagrangian Multiplier Method

This method is used to solve the optimization problems of a complex nature and those which cannot be solved by the substitution method. We will, however, illustrate the Lagrangian multiplier method in respect of

(i) a constrained profit maximization problem, and

(ii) a constrained cost minimization problem.

(i) Constrained Profit Maximization

Let us restate the problem as

Maximize $\pi = 100X - 2X^2 - XY + 180Y - 4Y^2$ .... (3.34)
Subject to the constraint $X + Y = 30$ .... (3.35)

The basic approach of the Lagrangian multiplier method is to form a Lagrangian function by combining the objective function and the constraint equation and then solve it **by the partial derivative method**. There is a simple technique of formulating the Lagrangian function. First, set the constraint equation (3.35) equal to zero, i.e.,

$$X + Y - 30 = 0$$

Second, multiply the resulting equation by λ (the Greek letter "lambda'), i.e.,

$$\lambda(X + Y - 30) = \lambda X + \lambda Y - \lambda 30$$

And, finally add the resulting equation to the objective function. Thus, the Lagrangian function is formed as

$$L\pi = 100X - 2X^2 - XY + 180Y - 4Y^2 + \lambda(X + Y - 30) \text{ .... } (3.36)$$

Equation (3.36) is the unconstrained Lagrangian function with three unknowns, X, Y and λ. The values of X, Y and λ that maximize $L\pi$ maximize π also: The Greek letter λ is the *Lagrangian multiplier*. **It gives the measure of a small change in the constraint on the objective function.**

What we need to do to maximize the $L\pi$ function (3.36) is to obtain the partial derivative of $L\pi$ with respect to X, Y and λ and set each of them equal to zero. This will give us the first order condition of profit maximization in the form of three simultaneous equations, as shown below.

$$\frac{\delta L\pi}{\delta X} = 100 - 4X - Y + \lambda = 0 \quad \text{........................... } (3.37)$$

$$\frac{\delta L\pi}{\delta Y} = -X + 180 - 8Y + \lambda = 0 \quad \text{........................... } (3.38)$$

$$\frac{\delta TC}{\delta \lambda} = X + Y - 30 = 0 \quad \text{........................... } (3.39)$$

By solving the simultaneous equations, we get the values of X, Y and λ that maximize the objective function (3.36). In order to solve these equations for X, Y and λ, we need to reduce the three simultaneous equations, (3.37), (3.38) and (3.39) to two equations. To do this, let us rearrange the terms of equations (3.38) and subtract it from equation (3.37). By subtracting we get,

$$\begin{array}{l} 100 - 4X - Y + \lambda = 0 \\ \underline{180 - X - 8Y + \lambda = 0} \\ -80 - 3X + 7Y \quad = 0 \text{ .... } (3.40) \end{array}$$

Now we have two simultaneous equations (3.40) and (3.39). Using the method of solving the simultaneous equations, we multiply equation (3.39) by 3 and add it to equation (3.40). Then we get,

$$3X + 3Y - 90 = 0$$
$$-\ \underline{3X + 7Y - 80 = 0}$$
$$10Y - 170 = 0$$
$$\underline{\underline{Y = 17}}$$

By substituting 17 for Y in the constraint equation (3.35), we get the value of X as

$$X + 17 = 30$$

or $$\underline{\underline{X = 13}}$$

Note that the values of X and Y are the same as computed above.

The value of λ can be obtained by substituting the values of X and Y in equation (3.37) or in equation (3.38). Using equation (3.38), we get,

$$-13 + 180 - 8\ (17) + \lambda = 0$$
$$\underline{\underline{\lambda = -31}}$$

The value of λ has an important economic interpretation. It gives the measure of the change in the total profit when the output constraint is changed by 1 unit. For example, if output is increased by 1 unit, i.e., from 30 to 31 units, profit will increase by about 31 and if output is decreased by 1 unit, i.e., from 30 to 29 units, the profit will decrease by about 31.

(ii) Constrained Cost Minimization: Suppose Josef Carpets, a carpet manufacturing and exporting firm, has to supply an order for 500 pieces of woolen carpets of two varieties X and Y to a German buyer. The joint cost function for the two varieties of carpets is given as

$$C = 100\ X^2 + 150\ Y^2 \quad \text{..........................} \quad (3.41)$$

The quantity of X and Y are not specified in the order. So the firm is free to supply X and Y in any combination. The firm's problem is to find the combination of X and Y that minimizes the cost of production subject to the constraint $X + Y = 500$. The problem can be restated formally as

Minimize $$C = 100\ X^2 + 150\ Y^2$$

Subject to $$X + Y = 500 \quad \text{..........................} \quad (3.42)$$

In order to solve the cost minimization problem by Lagrangian multiplier method, the problem has to be converted into a Lagrangian function. The procedure is to set the constraint equation (3.42) equal to zero, multiply it by λ and add the result to the objective function. The cost minimization problem converted into the Lagrangian function is given below.

Minimize $L_C = 100\,X^2 + 150\,Y^2 + \lambda\,(500 - X - Y)$ .... (3.43)

Subject to $500 - X - Y = 0$

The objective here is to minimize equation (3.43) subject to $X + Y = 500$. The first order condition of the solution requires that the derivative of L_C with respect to X, Y and λ is set equal to zero. Thus,

$$\frac{\delta L_C}{\delta X} = 200X - \lambda = 0 \quad \text{........................... (3.44)}$$

$$\frac{\delta L_C}{\delta Y} = 300Y - \lambda = 0 \quad \text{........................... (3.45)}$$

$$\frac{\delta L_C}{\delta \lambda} = 500 - X - \mathrm{Y} = 0 \quad \text{........................ (3.46)}$$

By subtracting equation (3.45) from equation (3.44), we get

$$200X - \lambda - (300\,Y - \lambda) = 0$$

$$200X = 300Y$$

$$\underline{\underline{X = 1.5Y}}$$

By substituting 1.5 Y for X in equation (3.46), we get

$$500 - 1.5\,Y - Y = 0$$

$$500 = 2.5Y$$

$$\underline{\underline{Y = 200}}$$

By substituting the value of Y with 200 in the constraint equation (3.42), we get

$$X + 200 = 500$$

$$\underline{\underline{X = 300}}$$

Thus, the solution to the cost minimization problem is that $X = 300$ and $Y = 200$ minimize the cost of producing 500 pieces of woolen carpets. The minimum cost can be worked out as follows

$$\begin{aligned} C &= 100\,X^2 + 150\,Y^2 \\ &= 100\,(300)^2 + 150\,(200)^2 \\ &= 9{,}000{,}000 + 6{,}000{,}000 \\ &= \underline{\underline{15{,}000{,}000}} \end{aligned}$$

Thus, the minimum cost of supplying 500 pieces if woolen carpets works out to Rs. 15 million. This is minimum cost because any other combination of X and Y varieties of carpets will make the cost exceed Rs. 15 million.